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APPLICATION OF VARIATIONAL METHODS TO THE THERMAL ENTRANCE REGION OF DUCTS

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Abstract—A variational method is presented for solving eigenvalue problems which arise in connection with the analysis of convective heat transfer in the thermal entrance region of ducts. Consideration is given to both situations where the temperature profile depends upon one cross-sectional co-ordinate (e.g. circular tube) or upon two cross-sectional co-ordinates (e.g. rectangular duct). The variational method is illustrated and verified by application to laminar heat transfer in a circular tube and a parallel-plate channel, and good agreement with existing numerical solutions is attained. Then, application is made to laminar heat transfer in a square duct as a check, an alternate computation for the square duct is made using a method indicated by Millsaps and Pohlhausen. The variational method can, in principle, also be applied to problems in turbulent heat transfer.

Résumé—Une méthode variationnelle est présentée pour la résolution des problèmes de valeurs propres qui se présentent dans l'analyse de la transmission de chaleur par convection dans la région d'entrée des conduites. On considère les deux cas dans lesquels le profil des températures dépend soit d'une seule coordonnée dans la section droite (c'est-à-dire tuyaux circulaires) ou de deux coordonnées (tuyaux rectangulaires). La méthode est vérifiée par application à la transmission de chaleur laminaire dans un tuyau circulaire et dans un canal à faces planes parallèles, on obtient un bon accord avec les solutions numériques existantes. Une application est faite ensuite à la transmission de chaleur laminaire dans une conduite carrée. A titre de contrôle, un calcul différent pour la conduite carrée est effectué en utilisant la méthode indiquée par Millsaps et Pohlhausen. Cette méthode variationnelle peut également être appliquée aux problèmes de transmission de chaleur turbulente.

Zusammenfassung—Zur Lösung von Eigenwertproblemen des konvektiven Wärmeübergangs beim thermischen Einlauf in Kanäle wird eine Variations-methode mitgeteilt. Es wird sowohl der Fall betrachtet, dass das Temperaturprofil von einer Querschnittskoordinate (z.B. Kreisrohr), als auch jener, bei der es von zwei Querschnittskoordinaten (rechtwinkliger Kanal) abhängt. Die Variationsmethode wird erläutert und auf den laminaren Wärmeübergang im Kreisrohr und im ebenen Spalt angewandt, wobei mit den bestehenden numerischen Lösungen eine gute Übereinstimmung gefunden wird. Sodann wird die Methode auf den laminaren Wärmeübergang im quadratischen Kanal angewendet und mit einer Iterationsrechnung nach Millsaps und Pohlhausen verglichen. Prinzipiell kann die Variations-methode auch auf Probleme des turbulenten Wärmeübergangs angewendet werden.

Abstract—Даётся вариационный метод решения задачи отыскания собственных значений уравнения, основанный на анализе конвективного теплообмена во входной области каналов. Рассматриваются два случая: один, когда температурный профиль зависит от одной координаты поперечного сечения (например, круглая труба), и второй — от двух координат поперечного сечения (например, канал прямоугольного профиля). Вариационный метод иллюстрируется и подтверждается применением его к случаям теплообмена при ламинарном потоке в круглой трубе и в плоско-параллельном канале; получается хорошее совпадение с существующими численными решениями. Затем, этот метод применяется к теплообмену в условиях ламинарного потока в канале с квадратным профилем. В последнем случае для контроля провели расчёт по указанному Миллсапсом и Польхаузеном методу. В принципе вариационный метод можно также применить и к задачам теплообмена в условиях турбулентного потока.

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NOMENCLATURE

A_{ni}, constants in eigenfunction equation (8);
a, half-height of parallel plate channel; half-length of side of square duct;
C_n, coefficients in temperature distribution;
c_p, specific heat at constant pressure;
d, tube diameter, $2r_0$;
e, f, g, functions of cross-section co-ordinates in equations (4a) and (10a);
I_0, I_1, I_2, integrals used in the variational procedure;
J, variational expression, $I_1 - \beta_n^2 I_2$;
k, thermal conductivity;
N, normal direction;
n, index number of eigenvalue;
Pr, Prandtl number, ν/α;
p, number of terms in eigenfunction equation (8);
Q, heat transfer rate per unit length;
q, heat transfer rate per unit area;
Re, tube Reynolds number, $\bar{w}d/\nu$;
Re_a, channel Reynolds number, $\bar{w}a/\nu$;
R_n, nth eigenfunction;
R_{ni}, functions in eigenfunction equation (8);
r, radial co-ordinate; r_0, tube radius;
T, static temperature; T_w, wall temperature; T_b, bulk temperature; T_0, entering temperature;
w, axial velocity distribution; $\bar{w}$, average velocity;
x, cross-section co-ordinate; $X = x/a$;
y, cross-section co-ordinate; $Y = y/a$;
z, axial co-ordinate; $Z = z/a$;
α, thermal diffusivity, $k/\rho c_p$;
β_n^2, nth eigenvalue;
η, η_1, η_2, cross-section co-ordinates;
ν, kinematic viscosity;
ρ, density;
ϕ, fully developed temperature distribution in square duct;

Subscript:
fd, denotes fully developed condition.

INTRODUCTION

In a previous paper [1], it was shown that variational methods could be successfully applied to the computation of the fully developed heat transfer characteristics for forced-convection flow in passages. Here, attention is directed to the thermal entrance region of flow passages. Our aim is to formulate and apply a variational procedure which may be used to determine entrance-region heat-transfer results. To illustrate the method and to establish confidence in its predictions, variational calculations are first carried out for the circular tube and the parallel-plate channel, and comparisons are made with exact (numerical) solutions available in the literature. Then, the variational method is applied to the square duct, for which there are no entrance-region calculations in the literature and for which an exact solution has not been possible.† In all the examples considered, the flow is laminar and fully developed, while the heat input is uniform along the length of the passage. However, the variational method may also be applied to the isothermal-wall boundary condition and to turbulent flow.

DESCRIPTION OF THE VARIATIONAL METHOD

General remarks

As early as 1885, it was demonstrated by Graetz [2] that the temperature solution in the thermal entrance region of a passage could be formulated as an eigenvalue problem. His analysis was concerned only with fully developed laminar flow in a circular tube having uniform wall temperature. Later, it was found [3–7] that the eigenvalue formulation would provide entrance-region results for both fully developed laminar and turbulent flows in circular tubes and parallel-plate channels for either uniform wall temperature or uniform wall heat flux. The same sort of formulation will apply to other non-circular ducts, as will be shown in a succeeding section.

† added in proof: A paper, which appeared after this work was completed, gives some approximate results for rectangular ducts with boundary conditions different from those treated here. This work by S. C. R. Dennis, A. McD. Mercer, and G. Poots, appeared in *Quart. Appl. Math.* 17, 285 (1959).

To illustrate the way in which eigenvalues arise in the entrance-region heat transfer analysis, consider the problem of laminar flow in a circular tube with uniform wall heat flux. As shown in [6], the longitudinal variation of the wall temperature corresponding to the prescribed wall heat flux q may be written as

$$\frac{T_w - T_0}{q r_0/k} = \frac{11}{24} + \frac{4z/r_0}{Re\,Pr} + \sum_{n=1}^{\infty} C_n R_n(r_0) \exp\left(-\frac{\beta_n^2}{Re\,Pr}\frac{z}{r_0}\right) \quad (1)$$

where T_0 is the temperature of the fluid entering the tube and z is the distance measured from the entrance of the heated section. In the expression, β_n^2 and R_n, respectively, represent the eigenvalues and eigenfunctions of the following equation:

$$\frac{r_0}{r}\frac{d}{d(r/r_0)}\left[\frac{r}{r_0}\frac{dR_n}{d(r/r_0)}\right] + R_n\beta_n^2\left[1-\left(\frac{r}{r_0}\right)^2\right] = 0 \quad (2)$$

$$dR_n/dr = 0 \text{ at } r = 0 \text{ and } r = r_0$$

Solutions of this homogeneous equation with homogeneous boundary conditions are possible only for a discrete, though infinite number of β_n^2 values. Each β_n^2 which permits a solution is called an eigenvalue, and the function R_n associated with the eigenvalue is called an eigenfunction. Equation (2) is a special case of the general Sturm–Liouville type. From Sturm–Liouville theory, it is noted in [6] that:

$$C_n = -\frac{\int_0^1 [r/r_0 - (r/r_0)^3]\,[(r/r_0)^2 - \frac{1}{4}(r/r_0)^4 - 7/24]\,R_n\,d(r/r_0)}{\int_0^1 [r/r_0 - (r/r_0)^3] R_n^2\,d(r/r_0)} \quad (3)$$

From this, it is seen that once the eigenfunctions R_n are known, the coefficients C_n of equation (1) may be evaluated. So it is apparent that a knowledge of the eigenvalues and eigenfunctions of equation (2) holds the key to the temperature solution (1). Since the β_n^2 are all positive, it is seen that the series contribution to equation (1) dies away for large values of z. Thus, the series is only important in the entrance region and consequently, the eigenvalue problem is of interest only in connection with entrance-region heat transfer results. For the uniform wall temperature problem, the knowledge of eigenvalues is not only necessary in the entrance region, but also in the fully developed domain.*

Our goal here is to present and apply a variational procedure for solving eigenvalue problems which may arise in connection with entrance-region heat transfer computations. The variational method is especially advantageous for the lower eigenvalues, and because of this, it serves to compliment the approximate techniques of [8–10] which work best for higher eigenvalues.

Consideration will be given to two general types of eigenvalue problems which may be of heat transfer interest. The first type arises when the temperature profile depends only on one cross-sectional co-ordinate, for example, the radial position in a circular tube or the distance from the center line of a parallel-plate channel. We will call this the one-dimensional eigenvalue problem. The second type arises when the temperature profile depends upon two cross-section co-ordinates, as in a rectangular duct. We will call this the two-dimensional eigenvalue problem. Separate presentations and examples will be given for each of the two types of problems.

The one-dimensional eigenvalue problem

We direct our attention to the Sturm–Liouville eigenvalue problem, which will include as special cases all heat transfer situations in which the temperature depends on only one cross-section co-ordinate. The Sturm–Liouville problem with which we are concerned is to find the eigenvalues† β_n^2 and eigenfunctions R_n of the following equation:

$$\frac{d}{d\eta}\left(e\frac{dR_n}{d\eta}\right) + fR_n + \beta_n g\,R_n = 0 \quad (4a)$$

with the condition that on the boundary surface

$$R_n = 0 \text{ or } dR_n/d\eta = 0 \quad (4b)$$

The functions e, f and g may depend upon the

* The first eigenvalue, at least, is necessary in the fully developed region.

† Only positive eigenvalues will arise in the problems of interest here.

independent variable η. Now, according to the calculus of variations (e.g. [11]), there corresponds to equation (4) the following variational expression J:

$$J \equiv I_1 - \beta_n^2 I_2 \quad (5)$$

$$I_1 = \int_a^b [e(dR_n/d\eta)^2 - fR_n^2]d\eta \quad (6)$$

$$I_2 = \int_a^b gR_n^2\, d\eta \quad (7)$$

The variational expression J has a very special property, namely, that it takes on a stationary value, i.e. $\delta J \equiv 0$, when evaluated using a β_n^2 and an R_n which satisfy equations (4a) and (4b). This characteristic suggests a procedure for obtaining approximate solutions for the eigenvalues and eigenfunctions of equations (4a) and (4b) by using the variational expression J. The procedure is based on the Ritz method. According to this approach, a set of functions R_{n1}, $R_{n2}, \ldots$ is selected, each of which satisfies the boundary condition (4b). With these, the nth eigenfunction R_n is written as

$$R_n = A_{n1}R_{n1} + A_{n2}R_{n2} + \ldots = \sum_{i=1}^{p} A_{ni}R_{ni} \quad (8)$$

where the $A_{n1}, A_{n2}, \ldots$ are constants which remain to be determined. As explained later in detail, these unknown constants are found by imposing the condition that J take on a stationary value. An additional condition which must be satisfied by the eigenfunctions is that they are mutually orthogonal with respect to the weighting function g, i.e.

$$I_0 = \int_a^b gR_nR_m\, d\eta = 0, \quad m = n-1, n-2, \ldots, 1 \quad (9)$$

Thus, for the nth eigenfunction, there are $n - 1$ orthogonality equations. As a consequence, it is necessary to have at least n terms in the eigenfunction expression (8) to insure that the variational procedure can be fully carried through. For each eigenfunction, the variational procedure may be reapplied successively using an increasing number of terms in the series (8) until convergence to a desired accuracy is achieved. The convergence is hastened by choosing the form of the R_{ni} in accordance with any intuition or knowledge one may have about the problem. We now turn to describing the detailed steps by which the variational procedure is applied.

First eigenvalue: We begin by writing an expression for R_1 in the form of equation (8), selecting each function R_{1i} in the series to satisfy the boundary conditions. With this, the integrals I_1 and I_2, equations (6) and (7), are evaluated, and the variational expression J, equation (5), is constructed. For the first eigenfunction, the orthogonality condition (9) need not be considered, since there are as yet no other eigenfunctions. To find a stationary value of J, the expression is differentiated successively with respect to each A_{1i} and each resulting equation is set equal to zero. This provides a set of p linear, homogeneous (right-sides equal zero), algebraic equations involving the $A_{11}, A_{12}, \ldots, A_{1p}$. A solution to such a homogeneous set is possible only if the determinant of the coefficients is equated to zero, and from this, there arises a polynomial of which β_1^2 is the smallest root. Then, returning to the p homogeneous algebraic equations, it is possible to find $p - 1$ of the constants A_{1i}. The last coefficient is found by some other condition such as normalizing, i.e. setting $I_2 = 1$, or else by assigning some arbitrary value to R_1 at some particular value of η. As has already been noted, the procedure can then be repeated using a larger number of terms in the series (8) until a desired accuracy is attained.

Higher eigenvalues: The variational method for higher eigenvalues follows a path similar to that outlined for the first eigenvalue, except that the orthogonality condition (9) must now be incorporated into the procedure. This condition requires that the eigenvalues and eigenfunctions be found in ascending order; since the computation for the nth eigenfunction R_n assumes that all preceeding eigenfunctions are known. By employing the orthogonality conditions (9), it is possible to solve for $n - 1$ of the A_{ni} in terms of all the others. Then, turning to the expression for J, these particular A_{ni} may be eliminated by direct substitution. Thus, the number of A_{ni} remaining in J is reduced to $p - (n - 1)$. The stationary value of J is found by differentiating successively with respect to each of the remaining A_{ni} and equating each of the resulting equations to zero. From these $p - (n - 1)$ linear homogeneous equations, the eigenvalue β_n^2 can be found in a manner identical to that described in connection with the computation of the first eigenvalue. Then, the constants A_{ni} are computed using these $p - (n - 1)$ equations and the orthogonality conditions. Because these equations are homogeneous, there will still be one undetermined A_{ni} and this can be found by imposing additional conditions as previously described. The process can then be repeated, taking additional terms in the series (8) until a suitable result for β_n^2 and R_n is achieved.

To help fix these ideas, illustrative examples will be given in a later section.

The two-dimensional eigenvalue problem

For the situation where the temperature depends upon two cross-sectional co-ordinates, the two-dimensional eigenvalue problems which lend themselves to solution by the variational technique are given by the following equation [12]:

$$\frac{\partial}{\partial \eta_1}\left(e\frac{\partial R_n}{\partial \eta_1}\right) + \frac{\partial}{\partial \eta_2}\left(e\frac{\partial R_n}{\partial \eta_2}\right) + fR_n + \beta_n^2 g R_n = 0 \tag{10a}$$

with the condition that on the bounding surface

$$R_n = 0 \quad \text{or} \quad \partial R_n/\partial N = 0 \tag{10b}$$

In this equation, the functions e, f and g may depend on both co-ordinates η_1 and η_2. In a manner parallel to the one-dimensional case, there exists a variational expression J which corresponds to the eigenvalue problem defined by equations (10a) and (10b). Rephrasing the findings of [12], J may be written as

$$J = I_1 - \beta_n^2 I_2 \tag{11}$$

$$I_1 = \iint_{\text{area}} \left[e\left\{\left(\frac{\partial R_n}{\partial \eta_1}\right)^2 + \left(\frac{\partial R_n}{\partial \eta_2}\right)^2\right\} - fR_n^2\right] d\eta_1\, d\eta_2 \tag{12}$$

$$I_2 = \iint_{\text{area}} gR_n^2\, d\eta_1\, d\eta_2 \tag{13}$$

As before, the variational expression has the particular property that it takes on a stationary value when evaluated with an eigenvalue and eigenfunction of equations (10a) and (10b). This characteristic serves as a basis of an approximate method for finding the β_n^2 and R_n which is essentially identical to that which has already been described for the one-dimensional case. In applying the procedure to the two-dimensional case, it is to be remembered that the functions R_{ni} which make up the series (8) may now depend upon the two co-ordinates η_1 and η_2, instead of on the single η as before. Also, for the two-dimensional situation, the orthogonality condition takes the form

$$I_0 = \iint_{\text{area}} gR_nR_m\, d\eta_1\, d\eta_2 = 0, \quad m = n-1, n-2, \ldots, 1 \tag{14}$$

With these modifications, the detailed directions for using the variational method in the one-dimensional problem also apply here, and hence they need not be repeated.

An entrance-region heat transfer computation for a square duct will be carried out in a later section, and this will aid in further clarifying the application of the variational procedure to the two-dimensional case.

APPLICATION TO A CIRCULAR TUBE

As a first illustration of the application of the variational procedure, we consider the problem of laminar flow in a circular tube with uniform wall heat flux. The solution for the wall temperature as a function of position along the tube length has already been given in equation (1), while the associated eigenvalue problem is defined by equation (2). Now, for the uniform heat flux problem, it is well known that the first eigenvalue β_1^2 is zero, while R_1 is a constant usually taken as unity. It is also easy to show from equation (3) that $C_1 = 0$. So, with β_1^2 and R_1 known, we turn our attention to finding the second eigenvalue β_2^2 and its corresponding eigenfunction R_2.

To apply the variational method, we first compare equation (2) with the general form (4) and find that

$$\eta = r/r_0, \quad e = \eta, \quad f = 0, \quad g = \eta(1 - \eta^2)$$

With this, the integrals I_0, I_1 and I_2 become

$$I_0 = \int_0^1 \eta(1 - \eta^2)R_nR_m\, d\eta = 0, \quad m = n-1, n-2, \ldots, 1 \tag{15}$$

$$I_1 = \int_0^1 \eta(dR_n/d\eta)^2\, d\eta,$$
$$I_2 = \int_0^1 \eta(1-\eta^2)R_n^2\, d\eta \qquad (16)$$

The next step is to write R_2 as a sum of terms in the form of equation (8). In selecting the functions R_{2i}, we take cognizance of the boundary conditions (2) and of the fact that the eigenfunctions can have both positive and negative lobes between $\eta = 0$ and $\eta = 1$. Functions which immediately suggest themselves for the role of R_{2i} are $\cos 0\pi\eta = 1$, $\cos \pi\eta$, $\cos 2\pi\eta$, etc. So as a first approximation for R_2 in the form (8), we write:

$$R_2 = A_{21} + A_{22}\cos \pi\eta \qquad (17)$$

Then, substituting in equations (15) and (16) and carrying out the integration yields:*

$$I_0 = \frac{A_{21}}{4} + A_{22}\left(\frac{1}{\pi^2} - \frac{12}{\pi^4}\right) = 0 \qquad (18a)$$

$$I_1 = \frac{A_{22}^2\,\pi^2}{4} \qquad (18b)$$

$$I_2 = \frac{A_{21}^2}{4} + 2A_{21}A_{22}\left(\frac{1}{\pi^2} - \frac{12}{\pi^4}\right) + \frac{A_{22}^2}{8}\left(1 - \frac{3}{\pi^2}\right) \qquad (18c)$$

Now, using equations (18b) and (18c), we can construct the variational expression

$$J = I_1 - \beta_2^2 I_2$$

Then, by the orthogonality condition (18a), A_{21} can be eliminated from J, giving

$$J = \frac{A_{22}^2\pi^2}{4} - \beta_2^2\left[-4A_{22}^2\left(\frac{12}{\pi^4} - \frac{1}{\pi^2}\right)^2 + \right.$$
$$\left. + \frac{A_{22}^2}{8}\left(1 - \frac{3}{\pi^2}\right)\right] \qquad (19)$$

To find the stationary value, we take $\partial J/\partial A_{22} = 0$, and from this it follows that

$$\beta_2^2 = 28{\cdot}997 \qquad (20)$$

Considering the simplicity of the approximation and the relative ease of computation, this result is in surprisingly good agreement with the exact value [6]:

$$\beta_2^2 = 25{\cdot}6796 \qquad (21)$$

* It is to be remembered that $R_{n-1} = R_1 =$ constant.

The constants A_{21} and A_{22} remain to be determined. From the condition $\partial J/\partial A_{22} = 0$. there is obtained an algebraic equation which tells us nothing about the constants. But, from the orthogonality condition (18a), we have the ratio of A_{21} to A_{22}. With this, the eigenfunction expression (17) becomes

$$R_2 = A_{22}(0{\cdot}087482 + \cos \pi\eta) \qquad (17a)$$

As expected in accordance with what has been said in the general presentation of the variational method, there remains one undetermined constant which must be found from other conditions. To facilitate comparison with [6], we impose the condition that $R_2(0) = 1$, and with this, equation (17a) becomes

$$R_2 = 0{\cdot}080445 + 0{\cdot}91956 \cos \pi\eta \qquad (17b)$$

The value of R at $r = r_0$ ($\eta = 1$) plays an important role in the wall-temperature computation, as may be seen from equation (1). The first approximation, equation (17b), gives a value $R_2(1) = -0{\cdot}83911$, as opposed to $-0{\cdot}49252$ from the exact solution. This comparison strongly suggests that a higher variational approximation be carried out for R_2.

As a logical refinement of equation (17), we add on a term $\cos 2\pi\eta$ and write

$$R_2 = A_{21} + A_{22}\cos \pi\eta + A_{23}\cos 2\pi\eta \qquad (22)$$

Proceeding as before, I_0, I_1 and I_2 are computed by integration of equation (22). From the orthogonality condition, $I_0 = 0$, there is obtained

$$A_{21} = 0{\cdot}087482\, A_{22} + 0{\cdot}303964\, A_{23} \qquad (23)$$

Using I_1 and I_2, the variational expression $J = I_1 - \beta_2^2 I_2$ is evaluated to be

$$J = 2{\cdot}46740\, A_{22} - 3{\cdot}55556\, A_{22}A_{23} +$$
$$+ 9{\cdot}86960 = A_{23}^2 -$$
$$- \beta_2^2[0{\cdot}0870045\, A_{22}^2 - 0{\cdot}0121336\, A_{22}A_{23} +$$
$$+ 0{\cdot}115501\, A_{23}^2 - 0{\cdot}25\, A_{21}^2] \qquad (24)$$

A_{21} can be eliminated from this expression by using equation (23), leaving only A_{22} and A_{23}. The stationary value of J is achieved by taking

$\partial J/\partial A_{22} = 0$, $\partial J/\partial A_{23} = 0$. From this, there results:

$$\left.\begin{aligned} \frac{\partial J}{\partial A_{22}} &= 0 \rightarrow (4{\cdot}93480 - 0{\cdot}170182\,\beta_2^2)A_{22} + \\ &+ (-3{\cdot}55556 + 0{\cdot}0254293\,\beta_2^2)A_{23} = 0 \\ \frac{\partial J}{\partial A_{23}} &= 0 \rightarrow (-3{\cdot}55556 + \\ &+ 0{\cdot}0254293\,\beta_2^2)A_{22} + (19{\cdot}7392 - \\ &- 0{\cdot}184805\,\beta_2^2)A_{23} = 0 \end{aligned}\right\} \quad (25)$$

A nontrivial solution of this pair of linear, homogeneous equations is possible only if

$$\begin{vmatrix} (4{\cdot}93480 - 0{\cdot}170182\,\beta_2^2) & (-3{\cdot}55556 + 0{\cdot}0254293\,\beta_2^2) \\ (-3{\cdot}55556 + 0{\cdot}0254293\,\beta_2^2) & (19{\cdot}7392 - 0{\cdot}184805\,\beta_2^2) \end{vmatrix} = 0$$

This determinant yields a quadratic equation for β_2^2, the smallest root* of which is

$$\beta_2^2 = 25{\cdot}6956 \quad (26)$$

This is in remarkably close agreement with the value 25·6796 from the exact solution. Now, turning to the determination of the constants A_{2i}, there is first obtained from either of equations (25) the ratio A_{23}/A_{22}. Next, from equation (23), the ratio A_{21}/A_{22} is found. Finally there is imposed the condition that $R_2(0) = 1$. With this information, all three constants may be calculated and the expression for R_2 becomes

$$R_2 = 0{\cdot}109207 + 0{\cdot}746309 \cos \pi\eta + 0{\cdot}144484 \cos 2\pi\eta \quad (22a)$$

The value of R_2 at the tube wall ($\eta = 1$) has been already noted as playing an important role in the heat transfer results. Equation (22a) gives a value $R_2(1) = -0{\cdot}49262$, which is in excellent agreement with the result $-0{\cdot}49252$ from the exact solution. In addition to $R_n(1)$, the solution for the wall temperature variation as given by equation (1) depends on the coefficients C_n. These constants can be computed from the quotient of integrals given in equation (3). Utilizing equation (22a) for R_2, the coefficient C_2 is computed as 0·40680, which deviates by only 0·8 per cent from the value 0·40348 given by the exact solution.

This illustration demonstrates that the variational procedure is capable of providing excellent heat transfer results. The high level of agreement indicates that there is no need to refine R_2 further. However, if an exact solution had not been available for comparison purposes, it would have been necessary to take a higher approximation for R_2 to establish the level of accuracy.

If desired, the variational method could now be applied to the computation of the next eigenfunction R_3. No essential changes in the method are necessary, except that the expression for R_3 in the form of equation (8) would likely contain additional cosine terms. Also, in the orthogonality condition (15), we would use equation (22a) for the known eigenfunction R_2, while R_1 is a constant. However, since our purpose in considering the circular tube has only been to illustrate the variational method, the computation for the higher eigenvalues will not be carried out.

* Since the value of β_2^2 associated with the exact solution is an absolute minimum, it is clear that the best result is achieved by selecting the smallest root.

APPLICATION TO A PARALLEL PLATE CHANNEL

As a second example which may serve to establish confidence in the variational procedure, we turn to the problem of laminar flow in a parallel-plate channel with uniform wall-heat flux. Corresponding to the prescribed heat flux, the variation of the wall temperature along the length is given by:

$$\frac{T_w - T_0}{qa/k} = \frac{17}{35} + \frac{z/a}{Re_a\,Pr} + \sum_{n=1}^{\infty} C_n\,R_n(a) \exp\left\{-\frac{\beta_n}{(3/2)\,Re_a\,Pr}\,\frac{z}{a}\right\} \quad (27)$$

The eigenvalues R_n and eigenfunctions β_n are found from the following homogeneous system:

$$\frac{d^2R_n}{d(y/a)^2} + R_n\beta_n^2\,[1 - (y/a)^2] = 0 \quad (28a)$$

$$dR_n/dy = 0 \text{ at } y = 0 \text{ and } y = a \quad (28b)$$

while the coefficients C_n are computed from the ratio:

$$C_n = - \frac{\int_0^1 [1-(y/a)^2][\frac{3}{4}(y/a)^2 - \frac{1}{8}(y/a)^4 - 39/280] R_n \, d(y/a)}{\int_0^1 [1-(y/a)^2] R_n^2 \, d(y/a)} \quad (29)$$

Here again is a problem ideally suited for the variational procedure. Comparing equation (28a) with the general one-dimensional form (4a), it is seen that

$$\eta = y/a, \; e = 1, \; f = 0, \; g = 1 - \eta^2 \quad (30)$$

and the integrals I_0, I_1 and I_2 become

$$I_0 = \int_0^1 (1-\eta^2) R_n R_m \, d\eta = 0, \quad m = n-1, n-2, \ldots, 1 \quad (31a)$$

$$I_1 = \int_0^1 (dR_n/d\eta)^2 \, d\eta, \quad I_2 = \int_0^1 (1-\eta^2) R_n^2 \, d\eta \quad (31b)$$

Since we are dealing with the case of uniform wall heat flux, it follows that $\beta_1^2 = 0$, R_1 = constant, and $C_1 = 0$. So, attention can immediately be directed to the second eigenvalue R_2. Drawing upon our experience with the circular tube, we start out by writing R_2 in the form of equation (22). Following through the operations as before, it is found that

$$\beta_2^2 = 18{\cdot}39, \quad R_2 = -0{\cdot}3157 + 1{\cdot}124 \cos \pi\eta + 0{\cdot}1924 \cos 2\pi\eta \quad (32)$$

The numerical solution of [7] yields a value of 18·38 for β_2^2. For $R_2(1)$, which is needed in the wall-temperature computation, [7] gives $-1{\cdot}27$ as compared to $-1{\cdot}25$ from the variational solution (32). No results for C_2 are provided in [7] and so this computation is omitted here.

From the excellent level of agreement which has been demonstrated, one can draw a real feeling of confidence in the utility of the variational procedure.

APPLICATION TO A SQUARE DUCT

For the previous examples which have been presented, there exist numerical solutions in the literature. These illustrations have been useful in establishing a feel for the accuracy which could be achieved by the variational method. Now, we turn to a situation for which there are no entrance-region calculations in the literature.

Consideration is given to a laminar flow in a square duct, Fig. 1, having a heat flux which is uniform both along the length and around the periphery. Since this physical problem is not treated elsewhere, we start from first principles.

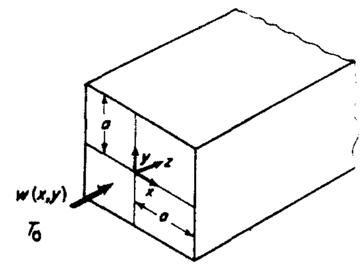

FIG. 1. Co-ordinate system for square duct.

The energy equation appropriate to the fully developed laminar flow of an incompressible, constant property fluid is

$$\rho c_p w \frac{\partial T}{\partial z} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (33)$$

where axial heat conduction has been neglected compared to transverse conduction. Far down the duct, $\partial T/\partial z$ becomes a constant and the condition of fully developed heat transfer is achieved. A solution for the temperature distribution in the fully developed regime has been found by variational means in [1] as follows.

$$\frac{T - T_0}{Q/8k} = \frac{2Z}{Re_a Pr} + \phi \quad (34a)$$

$$\phi = \frac{X^2 + Y^2}{2} - 0{\cdot}16032 \, [(X^2-1)^2 + (Y^2-1)^2] - 0{\cdot}13290 \, [(X^2-1)(Y^2-1)]^2 + 0{\cdot}06803 \quad (34b)$$

where T_0 is the temperature of the fluid entering the duct, Q is the heat transfer per unit length and X, Y and Z are dimensionless co-ordinates.

Now, turning to the entrance-region, we propose a solution in the following form which will apply anywhere throughout the duct:

$$\frac{T - T_0}{Q/8k} = \frac{2Z}{Re_a \, Pr} + \phi + \sum_{n=1}^{\infty} C_n R_n \exp\left(- \frac{\beta_n^2}{Re_a \, Pr} Z \right) \quad (35)$$

In this equation, R_n depends on both cross-section co-ordinates X and Y. To find the governing equation for R_n, (35) is introduced into the energy equation (33), from which it follows that:

$$\frac{\partial^2 R_n}{\partial X^2} + \frac{\partial^2 R_n}{\partial Y^2} + \beta_n^2(w/\bar{w})R_n = 0 \quad (36a)$$

It is also easy to show that:

$$dR/dX = 0 \text{ at } X \pm 1, \quad dR/dY = 0 \text{ at } Y = \pm 1 \quad (36b)$$

So, once again we have an eigenvalue problem. The first thought about solving equation (36) is to try separation of variables, i.e. to suppose that R_n is a product of a function of X and a function of Y. However, this approach is not successful since the velocity distribution $w/\bar{w}$ is not in a product form. Hence, it is necessary to deal with equation (36a) as it stands.

The eigenvalue problem as represented by equations (36a) and (36b) is well-suited to be attacked by the variational procedure. Comparing equation (36a) with the general two-dimensional form (10a), it is seen that:

$$\eta_1 = X,\ \eta_2 = Y,\ e = 1,\ f = 0,\ g = w/\bar{w} \quad (37)$$

With this, the integrals I_0, I_1 and I_2 as given by equations (14), (12) and (13) become

$$I_0 = 4 \int_0^1 \int_0^1 (w/\bar{w}) R_n R_m \, dX \, dY = 0, \quad m = n-1, n-2, \ldots, 1 \quad (38a)$$

$$I_1 = 4 \int_0^1 \int_0^1 [(\partial R_n/\partial X)^2 + (\partial R_n/\partial Y)^2] dX \, dY \quad (38b)$$

$$I_2 = 4 \int_0^1 \int_0^1 (w/\bar{w}) R_n^2 \, dX \, dY \quad (38c)$$

Since we are dealing with the uniform heat flux case, the first eigenvalue β_1^2 is zero, while $R_1 =$ constant and $C_1 = 0$. So consideration can be immediately given to the second eigenvalue R_2. In selecting a trial function for R_2, we are guided by prior experience with the parallel-plate channel. For that configuration, an eigenvalue expression in the form of equation (22) proved to be quite satisfactory. Generalizing equation (22) to the two-dimensional case, we write

$$R_2 = A_{21} + A_{22} \cos \pi X \cos \pi Y + A_{23} [\cos \pi X \cos 2\pi Y + \cos 2\pi X \cos \pi Y] \quad (39)$$

where the constants A_{21}, A_{22} and A_{23} remain to be determined by use of the variational method. The first step is to compute the I integrals of equations (38). To carry out the integrations, a knowledge of the velocity distribution $w/\bar{w}$ is required. An accurate velocity profile has been determined by variational means [1] as follows:

$$w/\bar{w} = (X^2 - 1)(Y^2 - 1)[2{\cdot}0983 + 0{\cdot}29181 (X^2 + Y^2) + 0{\cdot}87546 X^2 Y^2] \quad (40)$$

Utilizing this expression in conjunction with equation (39), the I integrals are found to be

$$I_0 = A_{21} + 0{\cdot}082665 A_{22} - 0{\cdot}0432577 A_{23} = 0 \quad (41a)$$

$$I_1 = 2\pi^2 [A_{22}^2 + 5A_{23}^2] \quad (41b)$$

$$I_2 = -4A_{21} + 0{\cdot}83989 A_{22}^2 + 2{\cdot}0001 A_{23}^2 + 1{\cdot}16410 A_{22}A_{23} \quad (41c)$$

Then, the variational expression $J = I_1 - \beta_2^2 I_2$ is constructed. Utilizing equation (41a), it is possible to eliminate A_{21} from J, leaving only A_{22} and A_{23}. The stationary value is found by setting $\partial J/\partial A_{22} = 0$ and $\partial J/\partial A_{23} = 0$. From this, there is obtained

$$\frac{\partial J}{\partial A_{22}} = 0 \rightarrow (39{\cdot}478 - 1{\cdot}6251 \beta_2^2)A_{22} - (1{\cdot}1927 \beta_2^2)A_{23} = 0 \quad (42a)$$

$$\frac{\partial J}{\partial A_{23}} = 0 \rightarrow (-1{\cdot}1927 \beta_2^2)A_{22} + (197{\cdot}39 - 3{\cdot}9852 \beta_2^2)A_{23} = 0 \quad (42b)$$

By setting the determinant of the coefficients equal to zero, there results a polynomial in β_2^2, the smallest root of which

$$\beta_2^2 = 20{\cdot}929 \quad (43)$$

provides the second eigenvalue. Next, the constants A_{21}, A_{22} and A_{23} are found by utilizing either of equations (42) in conjunction with (41a) plus an additional condition. In this instance, as a matter of variety, we will impose the condition that the eigenfunction be normalized, i.e. $I_2 = 1$. There is thus sufficient information to determine all the constants, and the final expression for the eigenfunction R_2 becomes:

$$R_2 = 0{\cdot}067685 - 0{\cdot}92477 \cos \pi X \cos \pi Y - - 0{\cdot}20252 [\cos \pi X \cos 2\pi Y + + \cos 2\pi X \cos \pi Y] \quad (44)$$

The last bit of information needed to compute temperature distribution results from equation (35) is the constant C_2. Imposing the condition that $T = T_0$ at the entrance section ($Z = 0$), equation (35) becomes

$$\sum_{n=1}^{\infty} C_n R_n = -\phi \quad (45a)$$

Then, multiplying through by $(w/\bar{w})R_m$ and integrating over X and Y from 0 to 1, it follows by use of the orthogonality condition (38a) that

$$C_n = -\frac{\int_0^1\int_0^1 (w/\bar{w}) R_n \phi \, dX \, dY}{\int_0^1\int_0^1 (w/\bar{w}) R_n^2 \, dX \, dY} \quad (45b)$$

Carrying out the integration for R_2, it is found that

$$C_2 = -0{\cdot}2467 \quad (46)$$

This completes the computation for the second eigenfunction. To check the level of accuracy of the results, the variational procedure might be reapplied to an expression for R_2 which contains additional terms. But, we have decided instead to redo the problem using a completely different approximation procedure. The method is a modification of an idea presented by Millsaps and Pohlhausen [13] and its application to the current problem is discussed briefly in the Appendix. The results of this alternate computation are in very good agreement with those of the variational method.

With the second eigenvalue at our disposal, we can now turn to a discussion of the heat transfer results. The temperature distribution corresponding to the prescribed heat flux is given by equation (35), where ϕ represents the fully developed solution as written in equation (34), while C_2, R_2 and β_2^2 are given by equations (46), (44) and (43), respectively. Since we have only one term of the series ($C_1 = 0$), attention must be directed to that portion of the duct near the fully developed region. When the heat flux is prescribed, the information of greatest interest is the resulting wall temperature. In this instance, where the heat flux is everywhere uniform, the wall temperature will vary around the periphery in any cross-section, as well as along the length. Because of the symmetry of the problem, consideration need only be given to a typical part of the wall: $X = 1$, $0 \leqslant Y \leqslant 1$. The wall-temperature result can be put in a convenient form by first introducing the bulk temperature T_b:

$$\frac{T_b - T_0}{Q/8k} = \frac{2Z}{Re_a Pr} \quad (47)$$

and then, it may be observed that under fully developed conditions

$$\frac{(T - T_b)_{fd}}{Q/8k} = \phi \quad (48)$$

With these, equation (35) can be rephrased as:

$$\frac{T - T_b}{(T - T_b)_{fd}} = 1 + \frac{\sum_{n=1}^{\infty} C_n R_n \exp\left(-\frac{\beta_n^2}{Re_a Pr} Z\right)}{\phi} \quad (49)$$

Finally, at the wall ($X = 1$, $0 \leqslant Y \leqslant 1$), there is obtained

$$\left.\begin{aligned} &\frac{T_w - T_b}{(T_w - T_b)_{fd}} = \\ &1 + C_2 \left[\frac{R_2}{\phi}\right]_{X=1} \exp\left(-\frac{\beta_2^2}{Re_a Pr} Z\right), \\ &0 \leqslant Y \leqslant 1 \end{aligned}\right\} \quad (50)$$

where the series has been truncated after $n = 2$. The group $C_2(R_2/\phi)_{X=1}$ as evaluated from the variational method has been plotted as a solid line on Fig. 2. In the region near the mid-wall ($Y = 0$), the group has negative values; while near the corner, the group has positive values. Using this information in conjunction with equation (50), it is seen that for locations near the corner, the wall-to-bulk temperature difference is greater in some part of the entrance-region than it is in the fully developed region.

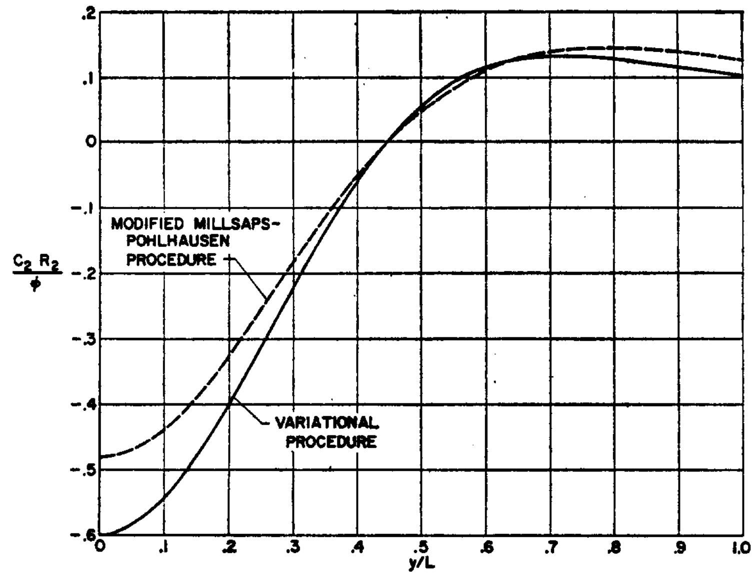

FIG. 2. Shape of temperature distribution along wall.

Of course, without additional terms in the series, it is not possible to know the extent of the entrance region where $(T_w - T_b) > (T_w - T_b)_{fd}$. This finding is somewhat surprising in view of previous experience with one-dimensional geometries such as the circular-tube and parallel-plate channel. In those configurations, where peripheral variations are absent, $T_w - T_b$ in the entrance-region is always less than the fully developed temperature difference. In the present instance, where there are peripheral variations, it is the feeling of the writers that $T_w - T_b$ is not the governing driving force for heat transfer at local positions around the periphery. Hence, there would be no reason to expect that the longitudinal variation of $T_w - T_b$ should follow the same trend here as was found in the simpler geometries. The fact that the longitudinal temperature variations are different in different parts of the cross-section suggests that there is little utility in computing average Nusselt numbers.

The findings derived from the variational method are closely confirmed by the alternate computation of the Appendix, as may be seen from the dashed line of Fig. 2. This good agreement increases our confidence in the results, but ultimate confirmation awaits some more exact solution, an approach to which is currently not apparent to the writers.

CONCLUDING REMARKS

The examples considered here are meant to illustrate the method and by no means exhaust the problems to which the variational technique can be applied.

Although the illustrations were concerned with the uniform heat flux case, solutions for uniform wall temperature are obtained with equal facility. In the latter case, the functions R_{ni} which make up the eigenfunction expression (8) would be selected to have zero values at the wall, rather than zero derivatives as in the former. It is also worth noting that the first eigenvalue for the uniform wall temperature situation will not be zero, nor will R_1 be a constant. Aside from these matters, the method is applied exactly as has been illustrated.

In principle, the variational procedure could be applied to problems in turbulent heat transfer. However, no work has thus far been done along these lines.

APPENDIX

Alternate Eigenvalue Calculation

According to the idea of Millsaps and Pohlhausen [13], the eigenfunctions for a laminar convective, heat transfer problem are expanded in terms of the eigenfunctions of the slug-flow problem. For the square duct, such an expansion for R_2 which is truncated after three terms is

$$R_2 = D_1 \cos \pi X \cos \pi Y + D_2(\cos \pi X \cos 2\pi Y + \cos 2\pi X \cos \pi Y) + D_3(\cos 2\pi X \cos 2\pi Y) \quad (51)$$

where D_1, D_2 and D_3 remain to be determined. Introducing this expression into the governing equation (36a) gives:

$$2\pi^2 D_1 \cos \pi X \cos \pi Y + 5\pi^2 D_2(\cos \pi X \cos 2\pi Y + \cos 2\pi X \cos \pi Y) + 8\pi^2 D_3(\cos 2\pi X \cos 2\pi Y) = \beta_2^2(w/\bar{w})R_2 \quad (52)$$

Then, equation (52) is multiplied through by $\cos \pi X \cos \pi Y$ and integrated over X and Y from 0 to 1. Next, equation (52) is multiplied by $(\cos \pi X \cos 2\pi Y + \cos 2\pi X \cos \pi Y)$ and integrated over the same range. Finally, this same procedure is carried through with $\cos 2\pi X \cos 2\pi Y$. The result of these operations is three linear, homogeneous, algebraic equations for the D_1, D_2 and D_3, the coefficients of which contain β_2^2. To obtain a nontrivial solution, the determinant of the coefficients is equated to zero, and this gives:

$$\beta_2^2 = 20{\cdot}222 \quad (53)$$

which is rather good agreement with equation (43) from the variational procedure. The D-values are then found by returning to the homogeneous algebraic equations, from which two of the three D's can be determined in terms of the third one. The third D value may be left unspecified or else arbitrarily assigned. This will have no effect on the final result for the desired quantity $C_n R_n$ (see equation (35)), since a change in the level of R_n will be automatically compensated by a corresponding change in C_n. Then, turning to the computation of C_2, the R_2 expression (with D-values now known) is introduced into equation (45b) and the integration carried out. With this, the final result for the $C_2 R_2$ product is

$$C_2 R_2 = 0{\cdot}2140 \cos \pi X \cos \pi Y + 0{\cdot}01740 \cos 2\pi X \cos 2\pi Y + 0{\cdot}04698(\cos \pi X \cos 2\pi Y + \cos 2\pi X \cos \pi Y) \quad (54)$$

Since the form of (54) is somewhat different from that of the variational eigenfunction, comparisons are best made by evaluating the expressions at specific X- and Y-values. Fig. 2 shows a comparison made along the wall, $X = 1, 0 \leqslant Y \leqslant 1$, with quite good agreement. A similar comparison has been made along the center-line in the fluid, $X = 0, 0 \leqslant Y \leqslant 1$, with about the same level of agreement.

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